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DECOMPOSING HYPERGRAPHS INTO SIMPLE HYPERTREES

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Let T be a simple k-uniform hypertree with t edges. It is shown that if H is any k-uniform hypergraph with n vertices and with minimum degree at least $\frac{n^{k-1}}{2^{k-1}(k-1)!}(1+o(1))$, and the number of edges of H is a multiple of t then H has a T-decomposition. This result is asymptotically best possible for all simple hypertrees with at least two edges.

1. Introduction

All hypergraphs and graphs considered here are finite and have no multiple edges. For the standard terminology used the reader is referred to [3]. Let H_1 and H_2 be two hypergraphs. We say that H_1 has an H_2-*decomposition* if the edge-set of H_1 can be partitioned into sets, such that the subhypergraph induced by each set is isomorphic to H_2. H_2 is called the *decomposing* hypergraph and H_1 is called the *decomposed* hypergraph. An obvious necessary condition for the existence of an H_2-decomposition is that $e(H_2)$ divides $e(H_1)$ ($e(X)$ and $v(X)$ denote, respectively, the number of edges and vertices of X).

The combinatorial and computational aspects of the decomposition problem have been studied extensively, especially in the graph-theoretic case. It is well-known, even for the simpler case of graphs, that the decomposition problem is NP-Complete even when the decomposing graph is fixed, and has a connected component with three or more edges [5]. Wilson, in a seminal result, has proved [7] that if H_1 is the complete graph K_n where

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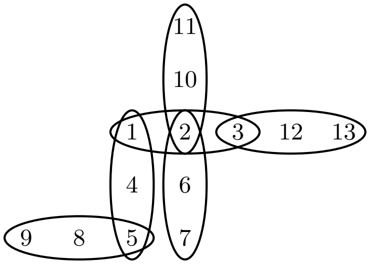

Fig. 1. A 3-uniform simple hypertree with 6 edges.

$n \geq n_0 = n_0(H_2)$, then K_n always has an H_2-decomposition, assuming the obvious necessary divisibility conditions hold. This author has proved in [8, 9] that when H_2 is a tree, Wilson's theorem holds not only when H_1 is a complete graph, but also when it has minimum degree $\lfloor v(H_1)/2 \rfloor$, and that this is best possible. There is no analogue of Wilson's theorem for Hypergraphs. Rödl [6] has shown how to obtain a large packing of the complete k-uniform hypergraph with smaller k-uniform hypergraphs (an H_2-packing of H_1 is a set of edge disjoint subhypergraphs of H_1 which are isomorphic to H_2). In fact, the only results on hypergraph decompositions are either constructions of small explicit designs (see [4] for many such constructions) or highly symmetric algebraic constructions ([1] is a good example). The result we present here is an analogue to the result of [8] for hypergraphs. As far as we know, this is the first hypergraph decomposition result where the decomposed hypergraph is general.

In order to describe our result we need several definitions. A *simple hyperforest* is a hypergraph possessing two properties:

1. Any two edges intersect in at most one vertex.
2. Every sequence of distinct edges $e_1, \ldots, e_r$ either have a common vertex, or there exists some j, $1 \leq j \leq r$, such that e_j and e_{j+1} are disjoint (we define $e_{r+1} = e_1$).

A connected simple hyperforest is called a *simple hypertree*. Note that a 2-uniform simple hyperforest is a forest (in the graph-theoretic sense), and a 2-uniform simple hypertree is a tree. Figure 1 is an example of a 3-uniform simple hypertree with 13 vertices and 6 edges. Our main theorem is the following:

Theorem 1.1. *Let T be a k-uniform simple hypertree with t edges, and let H be any k-uniform hypergraph with n vertices and tm edges where m is* *an integer. If the minimum degree of H satisfies*

$$\delta(H) \geq \binom{\lfloor n/2 \rfloor - 1}{k-1} + 9k^3t^5n^{k-4/3}$$

then H has a T-decomposition.

Note that, in particular, $\delta(H) \geq \frac{n^{k-1}}{2^{k-1}(k-1)!}(1+o(1))$.

The bound in theorem 1.1 is asymptotically tight for every simple hypertree with two edges or more (the decomposition problem is trivial when T has a single edge). We can construct a hypergraph H with tm edges and with minimum degree $\binom{\lfloor n/2 \rfloor - 1}{k-1} - 1$ which does not have a T-decomposition. Take two complete k-uniform hypergraphs H_1 and H_2 with orders $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$ respectively, and assume that $n \geq 2kt$. Delete $r_1 < t$ vertex-disjoint edges from H_1 such that the number of edges remaining in H_1 after the deletion is $1 \bmod t$. We can delete these edges since $\lfloor n/2 \rfloor \geq kt - 1 \geq r_1 k$. Now delete $r_2 < t$ vertex-disjoint edges from H_2 such that the number of edges remaining in H_2 after the deletion is $-1 \bmod t$. Now, if H is the disjoint union of H_1 and H_2 after these deletions, we have that H has $0 \bmod t$ edges, the minimum degree is $\binom{\lfloor n/2 \rfloor - 1}{k-1} - 1$, and, obviously, neither H_1 nor H_2 (after the deletions) have a T-decomposition, thus H does not have a T-decomposition.

Theorem 1.1 is, in fact, a corollary of a more general theorem, which states that hypergraphs with good edge expansion can be decomposed into simple hypertrees (assuming the divisibility condition holds). An n-vertex hypergraph is called r *edge-expanding* if for every subset of vertices X with $|X| \leq n/2$ there are at least $r|X|$ edges which contain a vertex from X and a vertex outside of X. The "expansion version" of Theorem 1.1 is:

Theorem 1.2. *Let T be a k-uniform simple hypertree with t edges, and let H be any k-uniform hypergraph with n vertices and tm edges where m is an integer. If H is $9k^2t^5n^{k-4/3}$ edge-expanding then H has a T-decomposition.*

Theorem 1.1 is an immediate corollary of Theorem 1.2 since it is easy to prove (cf. Lemma 4.8 in the last section) that a k-uniform hypergraph with minimum degree $\binom{\lfloor n/2 \rfloor - 1}{k-1} + (k-1)r$ is r edge-expanding.

In the following section we prove several lemmas which are necessary for the proof of Theorem 1.2. The proof is completed in Section 3. Section 4 contains some concluding remarks and open problems. Some ideas of the proof are similar to the proof of the graph-theoretic case in [8], but there are many additional obstacles appearing here which must be handled, and do not occur in the graph-theoretic case. This is not surprising since simple hypertrees are more complex objects than trees. Most of the proofs apply probabilistic arguments. Some of the lemmas are rather technical, so the reader is advised to first read the statements of all the lemmas and the various definitions, and only then read the proofs. Finally, we note that throughout this paper all the logarithms are natural.

2. Obtaining a homomorphic decomposition

For the rest of this paper, let T be a fixed k-uniform simple hypertree ($k \geq 2$) with $t > 1$ edges, and let $H = (V, E)$ be a k-uniform hypergraph with $|E| = tm$ edges where m is an integer and with $|V| = n$ vertices, which is also $9k^2t^5n^{k-4/3}$ edge-expanding. Our goal is to show that H has a T-decomposition. For each $v \in V$ we let $d(v)$ denote the degree of v in H.

Lemma 2.1. *E can be partitioned into t subsets $E_1, \dots, E_t$, such that $|E_i| = m$ and*

$$\left| d_i(v) - \frac{d(v)}{t} \right| \leq 3\sqrt{d(v) \log n},$$

where $d_i(v)$ is the number of elements of E_i containing v. Furthermore, each spanning subhypergraph $H_i = (V, E_i)$ is $k^2t^4n^{k-4/3}$ edge-expanding.

Proof. By definition, an r edge-expanding hypergraph must have minimum degree at least r, so for each $v \in V$ we have $d(v) \geq 9k^2t^5n^{k-4/3}$. In particular, $k|E| = ktm \geq 9k^2t^5n^{k-1/3}$. Thus,

$$m \geq 9kt^4n^{k-1/3} \tag{1}$$

and since, trivially, $n^{k-1} > d(v)$ we may also assume $n > 9^3k^6t^{15} \geq 9^3 \cdot 2^{21}$.

We let each edge $e \in E$ choose a random integer between 0 and t, where 0 is chosen with probability $\beta = n^{(1-k)/2}$ and the other numbers are chosen with probability $\alpha = (1-\beta)/t$. All the choices are independent. For $i = 0, \dots, t$, let $F_i \subset E$ be the set of edges which selected i. Let $d'_i(v)$ be the number of elements of F_i containing v. The expectation of $|F_i|$ is $\mu(|F_i|) = \alpha|E| = m(1-\beta)$, for $i \neq 0$. We apply a large deviation inequality attributed to Chernoff (cf., e.g. [2] Appendix A) to derive, using (1), that for $i \neq 0$

$$\text{Prob}[|F_i| > m] = \text{Prob}[|F_i| - \mu(|F_i|) > m\beta] < \exp\left(-\frac{2m^2\beta^2}{tm}\right) = \tag{2}$$

$$\exp\left(-\frac{2m}{n^{k-1}t}\right) \leq \exp\left(-\frac{18kt^3n^{k-1/3}}{n^{k-1}}\right) = \exp(-18kt^3n^{2/3}) < \frac{1}{n}.$$

Similarly, we have that for all $i \neq 0$ and for all $v \in V$

$$\text{Prob}[|d_i'(v) - \alpha d(v)| > \sqrt{d(v)\log n}] < 2\exp\left(\frac{-2d(v)\log n}{d(v)}\right) = \frac{2}{n^2}. \tag{3}$$

Analogously, for $i = 0$ we get

$$\text{Prob}[|d_0'(v) - \beta d(v)| > \sqrt{d(v)\log n}] < 2/n^2. \tag{4}$$

Since $n > 9^3k^6t^{15}$ we have

$$t \cdot (1/n) + nt \cdot (2/n^2) + n \cdot (2/n^2) = \frac{3t+2}{n} < 0.1.$$

Hence, we have by inequalities (2) (3) and (4) that with probability greater than 0.9, all of the following events hold simultaneously:

1. $|F_i| \leq m$ for $i = 1, \ldots, t$.
2. $|d_i'(v) - \alpha d(v)| \leq \sqrt{d(v)\log n}$ for all $i = 1, \ldots, t$ and for all $v \in V$.
3. $|d_0'(v) - \beta d(v)| \leq \sqrt{d(v)\log n}$ for all $v \in V$.

Consider, therefore, a partition of E into $F_0, \ldots, F_t$ in which all of these events hold. Since $|F_i| \leq m$, we may partition F_0 into t subsets $Q_1, \ldots, Q_t$, where $|Q_i| = m - |F_i|$. Put $E_i = F_i \cup Q_i$ for $i = 1, \ldots, t$. Note that $|E_i| = m$ and $E_i \cap E_j = \emptyset$ for $1 \leq i < j \leq t$. Put $H_i = (V, E_i)$ and let $d_i(v)$ be the degree of v in the hypergraph H_i. Clearly,

$$\begin{aligned} d_i(v) \geq d_i'(v) \geq \alpha d(v) - \sqrt{d(v)\log n} = \\ \frac{d(v)}{t} - \frac{d(v)}{n^{(k-1)/2}t} - \sqrt{d(v)\log n} \geq \\ \frac{d(v)}{t} - \frac{\sqrt{d(v)}}{t} - \sqrt{d(v)\log n} \geq \frac{d(v)}{t} - 2\sqrt{d(v)\log n}. \end{aligned} \tag{5}$$

We also need to bound $d_i(v)$ from above:

$$\begin{aligned} d_i(v) \leq d_i'(v) + d_0'(v) \leq \alpha d(v) + \beta d(v) + 2\sqrt{d(v)\log n} = \\ \frac{d(v)}{t} - \frac{d(v)}{n^{(k-1)/2}t} + 2\sqrt{d(v)\log n} + \frac{d(v)}{n^{(k-1)/2}} \leq \\ \frac{d(v)}{t} + 2\sqrt{d(v)\log n} + \frac{d(v)}{n^{(k-1)/2}} \leq \\ \frac{d(v)}{t} + 2\sqrt{d(v)\log n} + \sqrt{d(v)} \leq \frac{d(v)}{t} + 3\sqrt{d(v)\log n}. \end{aligned} \tag{6}$$

It now follows from inequalities (5) and (6) that $\left|d_i(v) - \frac{d(v)}{t}\right| \leq 3\sqrt{d(v)\log n}$.

Consider the partition of E into $F_0,\dots,F_t$. We have already shown that with probability greater than 0.9 this partition is good in the sense that one may obtain the desired partition into the subsets E_i by transferring vertices from F_0 to the F_i's. This, however, does not guarantee that the hypergraphs $H_i=(V,E_i)$ are $k^2t^4n^{k-4/3}$ edge-expanding, as required. Since edge-expansion is a monotone-increasing property, it suffices to show that with probability at least $1-0.9=0.1$, all of the hypergraphs $H_i'=(V,F_i)$ are $k^2t^4n^{k-4/3}$ edge-expanding. We prove this as follows: Let $X\subset V$ with $|X|\le n/2$. Let $n_i(X)$ denote the number of edges of F_i containing a vertex from X and a vertex outside of X. Our aim is to show that $n_i(X)\ge|X|k^2t^4n^{k-4/3}$, for all $i=1,\dots,t$ and for all X, with probability at least 0.1. Let $n(X)$ be the number of edges of H containing a vertex of X and a vertex outside of X. Since H is $9k^2t^5n^{k-4/3}$ edge-expanding we have that

$$n(X)\ge 9|X|k^2t^5n^{k-4/3}.$$

Clearly, $\mu(n_i(X))=\alpha n(X)$. Applying the large deviation bound once again we have

$$\text{Prob}[n_i(X)-\alpha n(X)<-\alpha n(X)/4]<\exp\left(-\frac{2n(X)^2\alpha^2/16}{n(X)}\right)=$$
$$\exp(-n(X)\alpha^2/8)\le\exp\left(-\frac{n(X)}{8(t+1)^2}\right)\le$$
$$\exp\left(-\frac{9|X|k^2t^5n^{k-4/3}}{8(t+1)^2}\right)\ll\frac{1}{nt\binom{n}{|X|}}$$

with lots of room to spare in the last part of this inequality. Since there are $\binom{n}{|X|}$ sets of cardinality $|X|$, and since there are $n/2$ possible cardinalities to consider, we get from the last inequality that with probability at least $0.5>0.1$, for all $i=1,\dots,t$ and for all sets $X\subset V$ with $|X|\le n/2$,

$$n_i(X)-\alpha n(X)\ge-\alpha n(X)/4.$$

In particular this means that

$$n_i(X)\ge\frac{3}{4}\alpha n(X)\ge\frac{3}{4}\cdot\frac{1}{t+1}9|X|k^2t^5n^{k-4/3}\ge|X|k^2t^4n^{k-4/3}. \qquad\blacksquare$$

From now on we fix a partition of E into subsets $E_1,\dots,E_t$ with the properties guaranteed by Lemma 2.1. Before we proceed to the next lemma, we need a few definitions. An *ordered hypergraph* is a hypergraph whose edges are ordered sets. For an edge e of an ordered hypergraph, we denote by $e(i)$ the vertex at position i in e. The first vertex in e is $e(1)$, which is called the *header* of e. We will assume in what follows that in each ordered edge, $e(i) \neq e(j)$ if $i \neq j$. The *underlying hypergraph* of an ordered hypergraph is the hypergraph obtained by ignoring the order. An *ordering* of a hypergraph X is an ordered hypergraph whose underlying hypergraph is X. The following lemma states a simple (and important) fact about simple hypertrees:

Lemma 2.2. *The edges of T can be labeled $e_1,\dots,e_t$ such that the following holds: There exists an ordering $\tilde{T}$ of T such that:*

1. *Each vertex appears as a nonheader in at most one edge.*
2. *For each $i > 1$, the header of e_i appears as a non-header in exactly one edge e_j, and $j < i$.*
3. *The header of e_1 never appears as a non-header of any other edge.*

Proof. A simple induction on t does the job. If $t = 1$ this is obvious. Otherwise, since T is a simple hypertree, there is some edge, which we shall denote by e_t, having the property that all vertices in e_t but one, have degree one in T. Order e_t such that the header is the vertex whose degree in T is greater than 1. This ordered edge, together with the ordering of the subtree on the other edges which exists by the induction hypothesis, is the desired ordering $\tilde{T}$. ∎

For example, if T is the tree depicted in Figure 1, we can let $\tilde{T}$ be $e_1 = (13,12,3)$, $e_2 = (3,2,1)$, $e_3 = (2,6,7)$, $e_4 = (2,10,11)$, $e_5 = (1,4,5)$, $e_6 = (5,8,9)$.

From now on we fix a labeling $e_1,\dots,e_t$ of T and an ordering $\tilde{T}$, with the properties of Lemma 2.2. Note that from the proof of Lemma 2.2 we have that e_1 can be chosen and ordered arbitrarily, so we may and will assume that the header of e_1 is a vertex of degree 1 in T (in the example above, the header of e_1 is 13, which, indeed, has degree one). Lemma 2.2 defines a parent-child relationship between the edges, where for each $i > 1$, the parent of e_i is the unique edge e_j which contains the header of e_i as a non-header. Let $p(i) = j$ if e_j is the parent of e_i, and note that $p(i) < i$. (In the example above, $p(2) = 1$, $p(3) = 2$, $p(4) = 2$, $p(5) = 2$, $p(6) = 5$.)

Our next lemma shows that it is possible to construct an ordering of H such that very precise requirements are met. We now describe the requirements from the ordering. Let $\tilde{H}$ be an ordering of H, and let $d_i(s,v)$ denote the number of edges of E_i, which contain v as the vertex at position s. Clearly $\sum_{s=1}^{k} d_i(s,v) = d_i(v)$, for all $v \in V$ and $i = 1,\dots,t$. Now let $i > 1$ and let s_i be the position of the header of e_i in $e_{p(i)}$. We require two properties from $\tilde{H}$:

Property 1. For each $v \in V$ and for each $i > 1$, $d_{p(i)}(s_i,v) = d_i(1,v)$.

Property 2. $\left|d_i(s,v)-\frac{d_i(v)}{k}\right| \le in^{k-4/3}$ for all $i=1,\dots,t$, $s=1,\dots,k$ and $v \in V$.

If both of these properties are met then $\tilde{H}$ is called a *T-homomorphic decomposition of H*. Note that the first property is very precise, we want the number of times v appears as a header in E_i to be equal to the number of times v appears at position s_i in $E_{p(i)}$. The reason for the appearance of i in the r.h.s. of the inequality in Property 2 will become apparent in the proof of the next lemma.

Lemma 2.3. *H has a T-homomorphic decomposition.*

Proof. We show how to construct $\tilde{H}$ in t stages, where in stage i we order the edges of E_i. We will show that Properties 1 and 2 in the definition of a T-homomorphic decomposition are met by E_i. We begin by ordering E_1. The ordering of each edge of E_1 is done randomly with uniform distribution, namely, each of the $k!$ possible orders has the same probability. All the m choices are independent. Condition 1 is empty for E_1, so we only need to show that Condition 2 holds with positive probability, and this is, once again, shown by a large deviation inequality. Using the fact that $\mu(d_1(s,v))=\frac{d_1(v)}{k}$ we have:

$$\text{Prob}\left[\left|d_1(s,v)-\frac{d_1(v)}{k}\right| > n^{k-4/3}\right] < 2\exp\left(-\frac{2n^{2k-8/3}}{d_1(v)}\right) < 2\exp\left(-\frac{2n^{2k-8/3}}{n^{k-1}}\right) < 2\exp(-2n^{1/3}) < \frac{1}{n^2}.$$

Now, since $k \cdot n \cdot \frac{1}{n^2} < 1$ we have that with positive probability, Property 2 holds for E_1. We therefore fix an ordering of E_1 which satisfies Property 2. Assume by induction that we have already ordered all the subsets E_j for $1 \le j < i$, such that Properties 1 and 2 hold for each E_j. We must show how to order the edges of E_i such that properties 1 and 2 also hold for E_i. Let $j=p(i)$, and recall that $j<i$. Also recall that s_i is the position of the header of e_i in e_j. Let $c_v = d_j(s_i,v)$. Note that c_v is already determined since E_j is already ordered. Property 1 implies that our ordering of E_i must satisfy that for each $v \in V$, $d_i(1,v)=c_v$. Hence, our initial task is to determine the headers of each edge of E_i such that this equality holds. Note that $\sum_{v\in V} c_v = m$. Thus, we must show that each vertex v can select c_v edges out of the $d_i(v)$ edges of E_i containing v, and such that each edge of E_i is selected by exactly one of its elements (if v selected an edge $f \in E_i$ then v becomes the header of f). In other words, we have a bipartite matching problem. Define a bipartite graph

B as follows: B has two vertex classes of order m each. One vertex class is E_i, while the other vertex class, denoted by S, contains c_v copies of each v. Thus, $S=\{v_w \mid v \in V, 1 \le w \le c_v\}$. The edges of B are defined as follows. An element $v_w \in S$ is connected to $e \in E_i$ if v is an element of e. Our goal is, therefore, to show that B has a perfect matching. By Hall's Theorem (cf. [3]), it suffices to show that for every set $S' \subset S$, $|N(S')| \ge |S'|$ where $N(S') \subset E_i$ are the neighbors of S' in B. Fix $\emptyset \ne S' \subset S$. Let $V'=\{v \in V \mid v_w \in S'\}$. Put $V'=\{v_1,\ldots,v_z\}$. Clearly, $|S'| \le \sum_{l=1}^{z} c_{v_l}$. Note that $N(S')$ is the set of edges of E_i which contain an element of V'. Let $Q_y \subset E_i$ be the set of edges having exactly y elements in V'. Clearly, $Q_1 \cup \ldots \cup Q_k = N(S')$. Put $q_i=|Q_i|$. Clearly, $q_1+2q_2+\ldots+kq_k=\sum_{l=1}^{z} d_i(v_l)$. We first consider the case $z \le n/2$. Since $H_i=(V,E_i)$ is $k^2t^4n^{k-4/3}$ edge-expanding and since $|V'|=z \le n/2$, we have

$$q_1+q_2+\ldots+q_{k-1} \ge zk^2t^4n^{k-4/3}. \tag{7}$$

Now, using Property 2 applied to $d_j(s_i,v_l)$ and using Lemma 2.1 we have:

$$\begin{aligned} \left|c_{v_l}-\frac{d_i(v_l)}{k}\right| &= \left|d_j(s_i,v_l)-\frac{d_i(v_l)}{k}\right| \le \\ \left|d_j(s_i,v_l)-\frac{d_j(v_l)}{k}\right| &+ \left|\frac{d_j(v_l)}{k}-\frac{d_i(v_l)}{k}\right| \le \\ jn^{k-4/3} &+ \frac{1}{k}\left(\left|d_j(v_l)-\frac{d(v_l)}{t}\right| + \left|d_i(v_l)-\frac{d(v_l)}{t}\right|\right) \le \\ jn^{k-4/3} &+ \frac{6\sqrt{d(v_l)\log n}}{k} \le \\ (t-1)n^{k-4/3} &+ 3\sqrt{d(v_l)\log n} \le t^2n^{k-4/3}. \end{aligned} \tag{8}$$

Thus, by (7) and (8) we have:

$$|N(S')| = q_1+\ldots+q_k = \sum_{l=1}^{z}\frac{d_i(v_l)}{k} + \frac{(k-1)q_1+(k-2)q_2+\ldots+q_{k-1}}{k} \ge$$
$$\left(\sum_{l=1}^{z}\frac{d_i(v_l)}{k}\right) + zkt^4n^{k-4/3} = \sum_{l=1}^{z}\left(\frac{d_i(v_l)}{k}+kt^4n^{k-4/3}\right) \ge \sum_{l=1}^{z} c_{v_l} \ge |S'|.$$

Now consider the case where $z > n/2$. Put $V''=V\setminus V'=\{v_{z+1},\ldots,v_n\}$. Note that $Q_1\cup\ldots\cup Q_{k-1}$ is the set of edges connecting V' with V''. Since H_i is $k^2t^4n^{k-4/3}$ edge-expanding and since $|V''| \le n/2$ we have $q_1+\ldots+q_{k-1} \ge (n-z)k^2t^4n^{k-4/3}$. Now,

$$|N(S')| = q_1+\ldots+q_k = \sum_{l=1}^{z}\frac{d_i(v_l)}{k} + \frac{(k-1)q_1+(k-2)q_2+\ldots+q_{k-1}}{k} \ge$$

$$\left(\sum_{l=1}^{z}\frac{d_i(v_l)}{k}\right)+(n-z)kt^4n^{k-4/3}=m-\sum_{l=z+1}^{n}\left(\frac{d_i(v_l)}{k}-kt^4n^{k-4/3}\right)\geq$$
$$m-\sum_{l=z+1}^{n}c_{v_l}=\sum_{l=1}^{z}c_{v_l}\geq|S'|.$$

We have shown that we can choose headers for the edges in E_i so that Property 1 holds. We now need to take care of Property 2. The case where $s=1$ can be shown immediately since $d_i(1,v)=c_v=d_j(s_i,v)$. Thus, by (8), and using the facts that $d(v)<n^{k-1}$ and $n>9^3\cdot 2^{21}$ we already have:

$$\left|d_i(1,v)-\frac{d_i(v)}{k}\right|\leq jn^{k-4/3}+\frac{6\sqrt{d(v)\log n}}{k}\leq(j+1)n^{k-4/3}\leq in^{k-4/3} \tag{9}$$

as required by Property 2. Each edge of E_i still has $k-1$ elements, except for the chosen header, which are still to be ordered. We shall order them randomly and uniformly (each of the $(k-1)!$ possible orders has the same probability), where all the m choices are independent. We show that the obtained ordering of E_i satisfies Property 2 with positive probability. The case $s=1$ was already treated. For $s>1$, note that the expectation of $d_i(s,v)$ is exactly $\mu(d_i(s,v))=\frac{d_i(v)-d_i(1,v)}{k-1}$. Thus, using the large deviation inequality, together with (9) we get

$$\mathrm{Prob}\left[\left|d_i(s,v)-\frac{d_i(v)}{k}\right|>in^{k-4/3}\right]=$$
$$\mathrm{Prob}\left[\left|d_i(s,v)-\mu(d_i(s,v))+\frac{1}{k-1}\left(\frac{d_i(v)}{k}-d_i(1,v)\right)\right|>in^{k-4/3}\right]<$$
$$\mathrm{Prob}\left[|d_i(s,v)-\mu(d_i(s,v))|>in^{k-4/3}-\left|\frac{1}{k-1}\left(\frac{d_i(v)}{k}-d_i(1,v)\right)\right|\right]<$$
$$\mathrm{Prob}\left[|d_i(s,v)-\mu(d_i(s,v))|>in^{k-4/3}-\frac{1}{k-1}in^{k-4/3}\right]<$$
$$\mathrm{Prob}[|d_i(s,v)-\mu(d_i(s,v))|>0.5in^{k-4/3}]<$$
$$\mathrm{Prob}[|d_i(s,v)-\mu(d_i(s,v))|>n^{k-4/3}]<2\exp\left(-\frac{2n^{2k-8/3}}{d_i(v)-d_i(1,v)}\right)<$$
$$2\exp\left(-\frac{2n^{2k-8/3}}{n^{k-1}}\right)<2\exp(-2n^{1/3})<\frac{1}{n^2}.$$

Since $(k-1)n\cdot\frac{1}{n^2}<1$, we have that with positive probability, for each $s>1$ and each $v\in V$, Property 2 holds for E_i. ∎

The following simple lemma supplies an absolute lower bound for $d_i(s,v)$ in a T-homomorphic decomposition:

Lemma 2.4. *$d_i(s,v) \ge 4kt^4n^{k-4/3}$ for all $i=1,\dots,t$, $s=1,\dots,k$ and $v \in V$.*

Proof. Using Property 2 of a T-homomorphic decomposition, and using Lemma 2.1 we have:

$$d_i(s,v) \ge \frac{d_i(v)}{k} - tn^{k-4/3} \ge \frac{d(v)}{kt} - \frac{3\sqrt{d(v)\log n}}{k} - tn^{k-4/3} \ge \frac{d(v)}{kt} - 4tn^{k-4/3} \ge 9kt^4n^{k-4/3} - 4tn^{k-4/3} \ge 4kt^4n^{k-4/3}.$$ ∎

3. Proof of the main result

A *homomorphism* between two hypergraphs H_1 and H_2 is a function $f: V(H_1) \to V(H_2)$ such that if $e \in E(H_1)$ then $f(e) \in E(H_2)$, and if $e \in E(H_2)$ then $f^{-1}(e) \in E(H_1)$. Note that if both H_1 and H_2 are k-uniform and have no isolated vertices, the definition implies that f is a surjection, and that f, restricted to each element of $E(H_1)$, is an injection. If f happens to be a bijection then it is called an *isomorphism*. With this definition, we can now show that a T-homomorphic decomposition $\tilde{H}$ of H (which exists by Lemma 2.3) defines a decomposition of the edges of H into a set L^* of m edge-disjoint hypergraphs, each having t edges, exactly one from each E_i. Furthermore, there is a homomorphism between T and each element of L^*. Unfortunately, these homomorphisms are not necessarily isomorphisms, so there is still some work to be done in order to obtain a decomposition of H into m copies of T.

We now describe the process which obtains L^* from $\tilde{H}$. Let $D_i(s,v)$ denote the set of edges of E_i in $\tilde{H}$, which have v at position s. Recall that $|D_i(s,v)| = d_i(s,v)$ and that $d_{p(i)}(s_i,v) = d_i(1,v)$. Therefore, there are $d_i(1,v)!$ ways to take a *perfect matching* between $D_{p(i)}(s_i,v)$ and $D_i(1,v)$. Let $B(i,v)$ be one such perfect matching, for each $i = 2,\dots,t$ and for each $v \in V$. The elements of $B(i,v)$ are, therefore, pairs of edges in the form (f_1,f_2) where $f_1 \in E_{p(i)}$ and $f_2 \in E_i$, the header of f_2 is v, and the vertex at position s_i in f_1 is also v. We say that f_1 and f_2 are *matched* if $(f_1,f_2) \in B(i,v)$ for some $i=2,\dots,t$ and some $v \in V$. Hence being matched is a symmetric relation on E. The transitive closure of the "matched" relation defines an equivalence relation where the equivalence classes are ordered subhypergraphs of $\tilde{H}$, each having t edges, one from each E_i, and which are homomorphic to $\tilde{T}$, by the homomorphism which maps a vertex of $\tilde{T}$ at position s in the edge e_i to the vertex at position s in the unique edge belonging to E_i in an equivalence class. Thus, L^* is the set of all of these subhypergraphs, (or, in set theoretical language, the quotient set of the equivalence relation). Note that although each $S \in L^*$ is homomorphic to $\tilde{T}$, it is not necessarily isomorphic to $\tilde{T}$ since S may contain cycles. Here is a simple example: Assume that $\tilde{T}$ is the 3-uniform simple hypertree defined by the edges $e_1 = (1,2,3)$ $e_2 = (2,4,5)$ and $e_3 = (5,6,7)$. It may be the case that some edge $(a,b,c) \in E_1$ is matched in $B(2,b)$ to the edge $(b,d,e) \in E_2$ and the edge (b,d,e) is matched in $B(3,e)$ to the edge (e,f,a). Hence, L^* contains an element S whose edges are $(a,b,c),(b,d,e)$ and (e,f,a). Obviously, S is homomorphic to $\tilde{T}$. However, this is not an isomorphism since both the vertices 1 and 7 are mapped to a, and, indeed, S is not a simple hypertree.

If we are lucky and all the m elements of L^* are isomorphic to $\tilde{T}$, we are done and have proved that H has a T-decomposition. This, however, may not be true, as demonstrated above. There are many ways to create L^*. In fact, since there are $d_i(1,v)!$ ways of picking each perfect matching $B(i,v)$ there are exactly

$$\Pi_{i=2}^{t} \Pi_{v \in V} d_i(1,v)!$$

different ways to create the decomposition L^*. Our goal is to show that in at least one of these decompositions, all the elements of L^* are, in fact, isomorphic to $\tilde{T}$, and this will conclude Theorem 1.2. Before proceeding with the proof of Theorem 1.2, we require a few definitions.

We say that the edge e_j of $\tilde{T}$ is a *descendant* of e_i if either $i = j$ or $e_{p(j)}$ is a descendant of e_i (note the recursive definition). Clearly, if $i > 1$, $\tilde{T}$ can always be partitioned into two nonempty simple hypertrees, one consisting of the descendants of e_i and the other consisting of the non-descendants of e_i. For an element $S \in L^*$, and for $i = 1,\dots,t$, let S_i be the subhypergraph of S which consists only of the first i edges, namely those belonging to $E_1 \cup \dots \cup E_i$. Let $S(i)$ be the edge of S belonging to E_i. Note that for $i > 1$, S_i is obtained from S_{i-1} by adding the edge $S(i)$. Trivially, S_1 is a simple hypertree consisting of one edge. Now, suppose S_{i-1} is a simple hypertree, and S_i is not a simple hypertree. This means that some non-header of the edge $S(i)$ already appears in S_{i-1}. Hence, we call an edge $S(i)$ *bad* if some non-header of $S(i)$ already appears in S_{i-1}. Otherwise, $S(i)$ is called *good*. Clearly, S is isomorphic to $\tilde{T}$ if and only if all its t edges are good. For $1 \le i \le j \le t$ and for $v \in V$ let us define the following set:

$$N(v,i,j) = \{S \in L^* \mid S(i) \in D_i(1,v) \text{ and } S(j) \text{ is bad}\}.$$

Clearly, $|N(v,i,j)| \le d_i(1,v)$. Our first goal is to show that if all the $n(t-1)$ perfect matchings $B(i,v)$ are selected randomly and independently, then with high probability, $|N(v,i,j)|$ is significantly smaller than $d_i(1,v)$. Before proving this, we need to state the following simple inequality which is proved using Stirling's Formula:

Lemma 3.5. *For every* $0<\alpha<1$ *and every positive integer* x*:*

$$\binom{x}{\lfloor x^\alpha \rfloor} < (e \cdot x^{1-\alpha})^{x^\alpha}$$

∎

Lemma 3.6. *If each of the perfect matchings* $B(i,v)$ *is selected randomly, and all the* $n(t-1)$ *choices are independent, then with probability at least* 0.9, *for all* $1 \le i \le j \le t$ *and for all* $v \in V$, $|N(v,i,j)| \le n^{k-4/3}$.

Proof. First note that by definition $N(v,1,1)=\emptyset$, so we may assume $j>1$. Since the perfect matchings are selected randomly and independently, we may assume that the n matchings $B_j(u)$ for all $u \in V$ are selected *after* all the other $n(t-2)$ matchings $B_r(u)$, for $r \neq j$, are selected. Prior to the selection of the last n matchings, the transitive closure of the "matched" relation defines two sets M^* and N^* each having m elements. Each element of M^* is a nonempty subhypergraph containing the edges of an equivalence class, with exactly one edge from each E_r where e_r is a descendant of e_j. Each element of N^* is a nonempty subhypergraph containing the edges of an equivalence class, with exactly one edge from each E_r where r is *not* a descendant of j. Note that the matchings $B_j(u)$ for all $u \in V$ match the elements of M^* with the elements of N^*, and each such match produces an element of L^*. Let us estimate $|N(v,i,j)|$ given that we know exactly what N^* contains; i.e. we shall estimate $\{|N(v,i,j)| \mid N^*\}$. Clearly, if $i<j$, then e_i is not a descendant of e_j and thus each element of N^* contains one edge from E_i. If $i=j$ then $p(i)<j$, and each element of N^* contains one edge from $E_{p(i)}$. We therefore call an element $S' \in N^*$ *relevant* if either $i<j$ and v is the header of the unique edge of S' belonging to E_i, or $i=j$ and v is the vertex at position s_i in the unique edge of S' belonging to $E_{p(i)}$. Note that, in any case, the number of relevant elements of N^* is exactly $d_i(1,v)$. Clearly, the value of $N(v,i,j)$ depends only on the relevant elements of N^* and on the perfect matchings $B_j(u)$ for each $u \in V$.

Consider a set $U=\{f_1,\dots,f_z\}$ of z edges of $E_{p(j)}$, where for $y=1,\dots,z$, each f_y belongs to some relevant element $S^{(y)}$ of N^* (indeed, since $p(j)<j$, $e_{p(j)}$ is not a descendant of e_j, so each relevant element of N^* contains one edge from $E_{p(j)}$, so we pick z such edges and call this set U). Let u_y be the vertex at position s_j in f_y. The perfect matching $B_j(u_y)$ matches f_y with an edge of E_j whose header is u_y (namely, an edge of $D_j(1,u_y)$). We call U *bad*, if for all $y=1,\dots,z$, f_y is matched in $B_j(u_y)$ to an edge which contains as a non-header some vertex which already appears in $S^{(y)}$ (in other words, it is matched to a bad edge). We wish to estimate the probability that U is bad. For this purpose, we must first estimate the number of edges in $D_j(1,u_y)$

which have some non-header already appearing in $S^{(y)}$. Consider some vertex $w \neq u_y$. Obviously, the co-degree of w and u_y in H (the co-degree of two vertices is the number of edges containing both of them) is at most n^{k-2}. Thus, the number of edges in $D_j(1,u_y)$ which contain as a non-header some vertex of $S^{(y)}$ is at most $(|S^{(y)}|-1)n^{k-2}$ where $|S^{(y)}|-1$ is the total number of vertices of $S^{(y)}$, not including u_y. Since $|S^{(y)}| \le kt$ we get that:

$$\text{Prob}[f_y \text{ is matched in } B_j(u_y) \text{ to a bad edge}] < \frac{ktn^{k-2}}{d_j(1,u_y)}.$$

Similarly, the probability that f_y is matched in $B_j(u_y)$ to a bad edge, *given that* $f_{y'}$ is matched in $B_j(u_{y'})$ to a bad edge, for all $1 \le y' < y$, is less than $ktn^{k-2}/(d_j(1,u_y)-(y-1))$. Thus,

$$\text{Prob}[U \text{ is bad}] < \Pi_{y=1}^{z} \frac{ktn^{k-2}}{d_j(1,u_y)-y+1}.$$

Now put

$$z = \lfloor d_i(1,v)^{\frac{k-4/3}{k-1}} \rfloor.$$

Note that $z < n^{k-4/3}$ so by Lemma 2.4, $z < d_j(1,u_y)/4$. Hence, again by Lemma 2.4:

$$\text{Prob}[U \text{ is bad}] < \left(\frac{ktn^{k-2}}{0.75d_j(1,u_y)}\right)^z \le \left(\frac{ktn^{k-2}}{3kt^4n^{k-4/3}}\right)^z \le \left(\frac{1}{3t^3}n^{-2/3}\right)^z.$$

Now, if there is no bad U, this means that, given N^*, $N(v,i,j) < z$. Thus, by considering all possible choices for U we obtain:

$$\text{Prob}[|N(v,i,j)| \ge z \mid N^*] < \binom{d_i(1,v)}{z}\left(\frac{1}{3t^3}n^{-2/3}\right)^z.$$

Note that the estimation in the last inequality does not depend on N^*, and thus,

$$\text{Prob}[|N(v,i,j)| \ge z] < \binom{d_i(1,v)}{z}\left(\frac{1}{3t^3}n^{-2/3}\right)^z.$$

We shall now use Lemma 3.5 with $x = d_i(1,v)$ and $\alpha = (k-4/3)/(k-1)$ and obtain:

$$\text{Prob}[|N(v,i,j)| \ge z] <$$

$$(e \cdot d_i(1,v)^{1/(3k-3)})^{d_i(1,v)^{\frac{k-4/3}{k-1}}} \left(\frac{1}{3t^3}n^{-2/3}\right)^{d_i(1,v)^{\frac{k-4/3}{k-1}}-1} \le$$

$$(e \cdot n^{1/3})^{d_i(1,v)^{\frac{k-4/3}{k-1}}} \left(\frac{1}{3t^3} n^{-2/3}\right)^{d_i(1,v)^{\frac{k-4/3}{k-1}}} 3t^3 n^{2/3} \leq$$

$$\left(\frac{e}{3t^3} n^{-1/3}\right)^{d_i(1,v)^{\frac{k-4/3}{k-1}}} 3t^3 n^{2/3} \ll \frac{1}{10t^2 n}.$$

Thus, with probability at least $1-n \cdot t^2 \frac{1}{10t^2 n} \geq 0.9$, for all $1 \leq i \leq j \leq t$, and for all $v \in V$, $|N(v,i,j)| < z < n^{k-4/3}$. ∎

The next lemma shows that, with high probability, any pair of distinct vertices do not appear together in elements of L^* too often.

Lemma 3.7. *If each of the perfect matchings $B(i,v)$ is selected randomly, and all the $n(t-1)$ choices are independent, then with probability at least $3/4$, for every two distinct vertices $u,w \in V$, the number of elements of L^* containing both u and w is at most $t^2 n^{k-4/3}$.*

Proof. Fix two distinct vertices u and w, and fix two integers i,j where $1 \leq i \leq j \leq t$. We shall estimate the number of elements $S \in L^*$ for which $u \in S(i)$ and $w \in S(j)$, and w is not the header of $S(j)$. We shall denote this number by $N(u,i,w,j)$. Now, in any element $S \in L^*$ which contains both u and w, at least one of these vertices is a non-header of some edge of S. This follows from Lemma 2.2 and from the fact that S is homomorphic to $\tilde{T}$. Furthermore, the only vertex in S which does not appear as a non-header in any edge of S, is the header of $S(1)$. It follows that the overall number of elements of L^* which contain both u and w, denoted by $N(u,w)$ satisfies:

$$N(u,w) \leq \sum_{i=1}^{t} \sum_{j=i}^{t} (N(u,i,w,j) + N(w,i,u,j)).$$

Thus, in order to prove the lemma it suffices to show that $N(u,i,w,j)$ is greater than $n^{k-4/3}$ with probability at most $1/(4n^2t^2)$.

Consider first the case where $i=j$. In this case, $N(u,i,w,j)$ is at most the co-degree of u and w and therefore $N(u,i,w,j) \leq n^{k-2} < n^{k-4/3}$, so the claim holds with probability 0 in this case.

We may now assume that $i<j$. Since the perfect matchings are selected randomly and independently, we may assume that the n matchings $B_j(v)$ for all $v \in V$ are selected *after* all the other $n(t-2)$ matchings $B_r(v)$, for $r \neq j$, are selected. Let N^* be defined as in the proof of Lemma 3.6. We shall estimate $N(u,i,w,j)$ *given* that we know N^*. Notice that since $i<j$ and $p(j)<j$, each of the m elements of N^* contains one edge from E_i and one edge from $E_{p(j)}$ (it may be that $i=p(j)$). Recall that s_j is the position

of the header of e_j in $e_{p(j)}$. For $v \in V$, Let F_v denote the set of elements of N^* for which u appears in the edge of N^* belonging to E_i, and v appears at position s_j in the edge of N^* belonging to $E_{p(j)}$. Put $f_v = |F_v|$. Clearly,

$$\sum_{v \in V} f_v = d_i(u).$$

Let G_v denote the set of edges of $D_j(1,v)$ which contain w, and let $g_v = |G_v|$. Consider the perfect matching $B_j(v)$. It is a perfect matching between $D_j(1,v)$ and the edges of $E_{p(j)}$ which have v at position s_j. Thus, the probability that a specific edge of $E_{p(j)}$ belonging to F_v is matched to an edge of G_v is exactly $g_v/d_j(1,v)$. We estimate $g_v/d_j(1,v)$ in case $v \neq w$ as follows: Clearly, g_v is at most the co-degree of v and w, and thus $g_v \leq n^{k-2}$. On the other hand, by Lemma 2.4, $d_j(1,v) > n^{k-4/3}$. Hence

$$\frac{g_v}{d_j(1,v)} \leq n^{-2/3} \quad \text{whenever} \quad v \neq w.$$

In order to estimate $N(u,i,w,j)$ given N^*, we shall define $N(u,i,w,j,v)$ for each $v \in V$, $v \neq w$ as follows: $N(u,i,w,j,v)$ is the number of elements of $E_{p(j)}$ belonging to F_v which are matched by the perfect matching $B_j(v)$ to an edge of G_v. Clearly:

$$[N(u,i,w,j) \mid N^*] = \sum_{v \in V\ ,\ v \neq w} N(u,i,w,j,v).$$

We shall therefore estimate $N(u,i,w,j)$ using estimates for $N(u,i,w,j,v)$. We must first dispose of the case where $k=2$, which must be treated differently (the graph-theoretic case). If $k=2$ then, trivially, $g_v \leq 1$ so $N(u,i,w,j,v) \leq 1$. In particular, $N(u,i,w,j,v) = 1$ if and only if (v,w) is an edge of $D_j(1,v)$ and in this case, by Lemma 2.4 we have

$$\text{Prob}[N(u,i,w,j,v) = 1] = \frac{f_v}{d_j(1,v)} \leq \frac{f_v}{n^{2/3}}.$$

Thus, the expectation of $N(u,i,w,j)$, given N^*, is exactly

$$\sum_{v \in V\ (v,w) \in D_j(1,v)} \frac{f_v}{d_j(1,v)} \leq \frac{d_i(u)}{n^{2/3}} \leq n^{1/3}.$$

Since the n perfect matchings $B_j(v)$ are chosen independently, we have that $N(u,i,w,j)$, given N^*, is the sum of at most n independent indicator random variables, so a large deviation inequality immediately gives:

$$\text{Prob}[N(u,i,w,j) - n^{1/3} > 0.5n^{2/3} \mid N^*] < \exp\left(-\frac{2 \cdot 0.25 \cdot n^{4/3}}{n}\right) < \frac{1}{4t^2n^2}.$$

Since the last inequality does not depend on N^* we have, in particular, that

$$\text{Prob}[N(u,i,w,j) > n^{2/3}] < \frac{1}{4t^2n^2}$$

as required.

We can now assume $k \geq 3$. We estimate $N(u,i,w,j,v)$ by considering two cases. If $f_v \leq n^{k-7/3}$ then we shall use the trivial estimate $N(u,i,w,j,v) \leq f_v \leq n^{k-7/3}$. If $f_v > n^{k-7/3}$ the estimate is done as follows. Let $0 < \alpha < 1$ be the real number which satisfies

$$f_v^{1-\alpha} = n^{1/3}.$$

Note that since $k \geq 3$ we have $f_v \geq n^{3-7/3} = n^{2/3}$, so α exists. Let $z = \lfloor f_v^\alpha \rfloor$. Let U be a set of exactly z edges of $E_{p(j)}$ taken from elements of F_v. U is called *bad* if all its elements are matched to edges of G_v. We shall prove that the probability that there exists a bad U is at most $1/(4n^3t^2)$. As shown above, the probability that each edge of U is matched to an edge of G_v is $g_v/d_j(1,v) \leq n^{-2/3}$. Hence, the probability that an edge of U is matched to an edge of G_v given that y previous edges of U are matched to edges of G_v is $(g_v - y)/(d_j(1,v) - y) \leq n^{-2/3}$. Thus,

$$\text{Prob}[U \text{ is bad}] \leq (n^{-2/3})^z.$$

Taking into account all possible sets U we get:

$$\text{Prob}[N(u,i,w,j,v) \geq z] \leq \binom{f_v}{z}(n^{-2/3})^z.$$

The last inequality is estimated using Lemma 3.5 with $x = f_v$ and α as defined above, giving:

$$\text{Prob}[N(u,i,w,j,v) \geq z] \leq (e \cdot f_v^{1-\alpha})^{f_v^\alpha}(n^{-2/3})^{f_v^\alpha} n^{2/3} = (en^{-1/3})^{f_v^\alpha} n^{2/3} \ll \frac{1}{4n^3t^2}.$$

Since $z \leq f_v^\alpha = f_v/n^{1/3}$ we have

$$\text{Prob}[N(u,i,w,j,v) \geq \frac{f_v}{n^{1/3}}] < \frac{1}{4n^3t^2}.$$

Hence, with probability at least $1 - \frac{1}{4n^2t^2}$, for each $v \in V$, we have $N(u,i,w,j,v) \leq \frac{f_v}{n^{1/3}}$. Hence, with probability at least $1 - \frac{1}{4n^2t^2}$:

$$[N(u,i,w,j) \mid N^*] = \sum_{v \in V\ ,\ v \neq w} N(u,i,w,j,v) \leq \sum_{w \neq v\ f_v \leq n^{k-7/3}} n^{k-7/3} + \sum_{w \neq v\ f_v > n^{k-7/3}} \frac{f_v}{n^{1/3}} \leq n^{k-4/3}.$$

Since the last estimate does not depend on N^* we have that $N(u,i,w,j) > n^{k-4/3}$ with probability at most $\frac{1}{4n^2t^2}$. ∎

Completing the proof of Theorem 1.2. According to Lemmas 3.6 and 3.7 we know that with probability at least 0.65, we can obtain a decomposition L^* with the properties guaranteed by Lemmas 3.6 and 3.7. We therefore fix such a decomposition, and denote it by L'. We let each element $S \in L'$ choose an integer $c(S)$, where $1 \le c(S) \le t$. Each value has equal probability $1/t$. All the m choices are independent. Let $C(v,i)$ be the set of elements $S \in L'$ for which $c(S) = i$ and for which v is the header of $S(i)$. Put $|C(v,i)| = c(v,i)$. Clearly, $0 \le c(v,i) \le d_i(1,v)$, and the expectation of $c(v,i)$ is $\mu(c(v,i)) = d_i(1,v)/t$. Since the choices are independent, we know that

$$\text{Prob}\left[c(v,i) < \frac{d_i(1,v)}{t+1}\right] < \exp\left(-\frac{2d_i(1,v)^2}{(t+1)^2t^2d_i(1,v)}\right) \le \exp\left(-\frac{2d_i(1,v)}{(t+1)^4}\right) < \frac{1}{2nt}.$$

Thus, with positive probability (in fact, with probability at least 0.5), we have that for all $v \in V$ and for all $i = 1,\dots,t$,

$$c(v,i) \ge \frac{d_i(1,v)}{t+1}. \tag{10}$$

We therefore fix the choices $c(S)$ for all $S \in L'$ such that (10) holds.

We are now ready to mend L' into a decomposition L consisting only of simple hypertrees isomorphic to $\tilde{T}$. Recall that each element of L' is homomorphic to $\tilde{T}$. We shall perform a process which, in each step, reduces the overall number of bad edges in L' by at least one, while maintaining the homomorphisms. Thus, at the end, there will be no bad edges, and all the elements are, therefore, simple hypertrees isomorphic to $\tilde{T}$. Our process uses two sets L_1 and L_2 where, initially, $L_1 = L'$ and $L_2 = \emptyset$. We shall maintain the invariant that, in each step in the process, $L_1 \cup L_2$ is a decomposition of H into subhypergraphs homomorphic to $\tilde{T}$. Note that this holds initially. We shall also maintain the invariant that $L_1 \subset L'$. Our process halts when no element of $L_1 \cup L_2$ contains a bad edge, and by putting $L = L_1 \cup L_2$ we obtain a decomposition of H into copies of $\tilde{T}$, as required. As long as there is some $S^\alpha \in L_1 \cup L_2$ which contains a bad edge, we show how to select an element $S^\beta \in L_1$, and how to create two subhypergraphs S^γ and S^δ which are also homomorphic to $\tilde{T}$ with $E(S^\alpha) \cup E(S^\beta) = E(S^\gamma) \cup E(S^\delta)$, such that the number of bad edges in $E(S^\gamma) \cup E(S^\delta)$ is *less* than the number of bad edges in $E(S^\alpha) \cup E(S^\beta)$. Thus, by deleting S^α and S^β from $L_1 \cup L_2$ and inserting

S^γ and S^δ both into L_2, we see that $L_1 \cup L_2$ is a *better* decomposition since it has less bad edges. It remains to show that this procedure can, indeed, be completed.

Let i be the maximum integer such that there exists an element $S^\alpha \in L_1 \cup L_2$ where $S^\alpha(i)$ is bad. Let v be the header of $S^\alpha(i)$. Consider the subhypergraph S^ϵ of S^α consisting of all the edges $S^\alpha(j)$ where e_j is a descendant of e_i. Our aim is to find an element $S^\beta \in L_1$, which satisfies the following requirements:

1. $c(S^\beta) = i$.
2. The header of $S^\beta(i)$ is v.
3. No vertex of S^α, except v, appears in S^β.

We show that an S^β meeting these requirements can always be found. The set $C(v,i)$ is exactly the set of elements of L' which meet the first two requirements (although some of them may not be elements of L_1). Let U be the set of vertices of S^α, not including v. For $u \in U$, all the elements of L' which contain both u and v are not allowed to be candidates for S^β, since this would violate the third requirement. According to Lemma 3.7, there are at most $t^2 n^{k-4/3}$ elements of L' containing both u and v. Let $C'(v,i)$ be the number of elements of $C(v,i)$ which satisfy the third requirement. Since U contains less than kt vertices, we have by (8), Lemma 2.4 and Lemma 3.7 that:

$$|C'(v,i)| \geq c(v,i) - kt^3 n^{k-4/3} \geq \frac{d_i(1,v)}{t+1} - kt^3 n^{k-4/3} > kt^3 n^{k-4/3}.$$

We need to show that at least one of the elements of $C'(v,i)$ is also in L_1. Each element $S \in C(v,i)$ that was removed from L' in a prior stage was removed either because it had a bad edge $S(j)$ where $j \geq i$ (this is due to the maximality of i), or because it was chosen as an S^β counterpart of some prior S^α, having a bad edge $S^\alpha(i)$ whose header is v. There are at most $|N(v,i,j)|$ elements $S \in C(v,i)$ which have a bad edge $S(j)$ where $j \geq i$, and there are at most $|N(v,i,i)|$ elements $S \in C(v,i)$ having $S(i)$ as a bad edge. According to Lemma 3.6, $\sum_{j=i}^{t} |N(v,i,j)| + |N(v,i,i)| \leq (t+1)n^{k-4/3} < |C'(v,i)|$. Thus, we have shown that the desired S^β can be selected.

Let S^π be the subhypergraph of S^β consisting of all the edges $S^\beta(j)$ where e_j is a descendant of e_i. S^γ is defined by taking S^α and replacing its subhypergraph S^ϵ with the subgraph S^π. Likewise, S^δ is defined by taking S^β and replacing its subgraph S^π with the subgraph S^ϵ. Note that S^γ and S^δ are both still homomorphic to $\tilde{T}$, and that $E(S^\alpha) \cup E(S^\beta) = E(S^\gamma) \cup E(S^\delta)$, so by deleting S^α and S^β from $L_1 \cup L_2$, and by inserting S^γ and S^δ to L_2 we have that $L_1 \cup L_2$ is still a valid decomposition into subhypergraphs homomorphic to $\tilde{T}$. The crucial point however, is that every edge of $E(S^\alpha)\cup E(S^\beta)$ that was good, remains good due to requirement 3 from S^β, and that the edge $S^\alpha(i)$ which was bad, now plays the role of $S^\delta(i)$, and it is now a good edge due to requirement 3. Thus, the overall number of bad edges in $L_1\cup L_2$ is reduced by at least one. ∎

4. Concluding remarks and open problems

1. As mentioned in the introduction, theorem 1.1 is an immediate consequence of Theorem 1.2, since every k-uniform hypergraph with minimum degree $\binom{\lfloor n/2\rfloor-1}{k-1}+(k-1)r$ is r edge-expanding. This is formally proved as follows:

Lemma 4.8. *Let H be a k-uniform hypergraph with minimum degree $\binom{\lfloor n/2\rfloor-1}{k-1}+(k-1)r$. Then H is r edge-expanding.*

Proof. Let X be any subset of vertices with $|X|\le n/2$. For each $v\in X$ there are at most $\binom{|X|-1}{k-1}\le\binom{\lfloor n/2\rfloor-1}{k-1}$ edges containing v and which lie completely within X. Hence, there are at least $(k-1)r$ edges containing v and at least one vertex outside of X. Summing up for each $v\in X$ there is a total of at least $(k-1)r|X|$ such edges. Each such edge is counted at most $k-1$ times (once for each vertex it contains from X). Hence, there are at least $r|X|$ edges containing a vertex of X and a vertex outside of X. ∎

2. The proof of Theorem 1.2 can also be implemented as a randomized algorithm. That is, given an input hypergraph H satisfying the edge-expansion properties, and with tm edges where m is an integer, one can produce a T-decomposition of H with constant positive probability. To see this, note that Lemma 2.1 is algorithmic, as the partition into the F_i's having the required properties can be done with probability of success at least 0.9, and the F_i's can be checked to have the required properties in polynomial time. The correction of the F_i's into the subsets E_i can be done in polynomial time with probability of success at least 0.5. The ordering of the edges satisfying the properties of a T-homomorphic decomposition can be performed by using any polynomial time algorithm for bipartite matching. After choosing the $n(t-1)$ perfect matchings $B(i,v)$ randomly and independently, one can compute in polynomial time that the obtained L^* satisfies the conditions in Lemmas 3.6 and 3.7. This happens with probability at least 0.65, according to these lemmas. If this is the case, the choices for $C(v,i)$ in Theorem 1.2 can be checked to comply with (10) in polynomial time, and (10) holds with probability at least 0.5. The final step of mending L' into the desired decomposition L is a purely sequential, non-randomized process, which can be done in polynomial time.

3. The power $k-4/3$ appearing in Theorem 1.1 and in Theorem 1.2 can somewhat be improved, but we are currently unable to get rid of the dependency on k. We conjecture, however, that this is possible in Theorem 1.1, and even more so:

Conjecture 4.1. For every k-uniform simple hypertree T with $t>1$ edges, there exists a constant $c(T)$ such that every k-uniform hypergraph H with tm edges, where m is an integer, and with

$$\delta(H) \geq \binom{\lfloor n/2 \rfloor - 1}{k-1} + c(T)$$

has a T-decomposition.

4. Although Theorem 1.1 and Theorem 1.2 are stated for k-uniform simple hypertrees, it is obvious that the theorems and all the lemmas used to prove them, can be easily modified to hold in case the decomposing hypergraph is a k-uniform simple hyperforest.

References

[1] N. Alon: Decomposition of the complete r-graph into complete r-partite r-graphs, *Graphs and Combinatorics,* **2** (1986), 95–100.

[2] N. Alon and J. H. Spencer: *The Probabilistic Method*, John Wiley and Sons Inc., New York, 1991.

[3] C. Berge: *Graph and Hypergraphs*, North-Holland, Amsterdam, 1973.

[4] C. J. Colbourn and J. H. Dinitz: *CRC Handbook of Combinatorial Design*, CRC press 1996.

[5] D. Dor and M. Tarsi: Graph decomposition is NPC—A complete proof of Holyer's conjecture, *Proc. 20th ACM STOC,* ACM Press (1992), 252–263.

[6] V. Rödl: On a packing and covering problem, *European J. Combinatorics,* **5** (1985), 69–78.

[7] R. M. Wilson: Decomposition of complete graphs into subgraphs isomorphic to a given graph, *Congressus Numerantium,* **XV** (1975), 647–659.

[8] R. Yuster: Tree decomposition of graphs, *Random Structures and Algorithms,* **12** (1998), 237–251.

[9] R. Yuster: Packing and decomposition of graphs with trees, *J. Combin. Theory Ser. B,* to appear.

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